Math 351 Notes

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Sequences and Series

UPPER AND LOWER LIMITS

So far we know that monotone, bounded sequences converge, and that any convergent sequence is necessarily bounded. These two facts together raise the question: Does every bounded sequence converge? Of course not. But just how "far" from convergent is a typical bounded sequence? To answer this, we want to broaden our definition of limits.

Observation: Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and consider the sequences:

$$t_n = \inf\{a_n,\ a_{n+1},\ a_{n+2},\ \ldots\} \quad \text{and} \quad T_n = \sup\{a_n,\ a_{n+1},\ a_{n+2},\ \ldots\}.$$

Then $\{t_n\}$ increases, $\{T_n\}$ decreases, and $\inf\limits_{k\in\mathbb{N}} a_k \le t_n \le T_n \le \sup\limits_{k\in\mathbb{N}} a_k$ for all n (why?). Thus we may speak of $\lim\limits_{n\to\infty} t_n$ as the "lower limit" and $\lim\limits_{n\to\infty} T_n$ as the "upper limit" of our original sequence $\{a_n\}$.

Now these same considerations are meaningful even if we start with an unbounded sequence $\{a_n\}$, although in that case we will have to allow the values $\pm\infty$ for at least some of the t_n's or T_n's (possibly both). That is, if we permit comparisons to $\pm\infty$, then the t_n's will increase and the T_n's will decrease. Of course we will want to use $\sup\limits_{n\in\mathbb{N}} t_n$ and $\inf\limits_{n\in\mathbb{N}} T_n$ in place of $\lim\limits_{n\to\infty} t_n$ and $\lim\limits_{n\to\infty} T_n$, since "sup" and "inf" have more or less obvious extensions to subsets of the extended real number system $[-\infty,\ \infty]$, whereas "lim" does not.

Even so, we are sure to get caught saying something like "$\{t_n\}$ converges to $+\infty$". But we will pay a stiff penalty for too much rigor here; even a simple fact could have a tediously long description. Therefore, for the remainder of this chapter we will interpret words such as "limit" and "converges" in this looser sense.

Definition: Given any sequence of real numbers $\{a_n\}$, we define

$$\lim_{n\to\infty} \inf a_n = \underline{\lim}_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k = \sup_{n\geq 1} \{\inf \{a_n, a_{n+1}, a_{n+2}, \ldots\}\}$$

and

$$\lim_{n\to\infty} \sup a_n = \overline{\lim}_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k = \inf_{n\geq 1} \{\sup \{a_n, a_{n+1}, a_{n+2}, \ldots\}\}$$

That is,

$$\lim_{n\to\infty} \inf a_n = \sup_{n\in\mathbb{N}} t_n \ \left(= \lim_{n\to\infty} t_n \text{ if } \{a_n\} \text{ is bounded from below}\right)$$

and

$$\lim_{n\to\infty} \sup a_n = \inf_{n\in\mathbb{N}} T_n \ \left(= \lim_{n\to\infty} T_n \text{ if } \{a_n\} \text{ is bounded from above}\right).$$

The name "lim inf" is short for "limit inferior" while "lim sup" is short for "limit superior".

Example:

a) Let $a_n = \frac{1}{n}$. Then $t_n = \inf_{k\geq n} a_k = 0$ and $T_n = \sup_{k\geq n} a_k = \frac{1}{n}$. Clearly $\lim_{n\to\infty} t_n = \lim_{n\to\infty} (0) = 0$ and $\lim_{n\to\infty} T_n = 0$. Thus $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n = 0$.

b) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $\left\{1, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, \ldots\right\}$. That is, $a_{2k} = \frac{k-1}{k}$ and $a_{2k-1} = \frac{1}{k}$. Then $t_n = 0$ and $T_n = 1$ (why?). Clearly $\lim_{n\to\infty} \inf(a_n) = 0 < 1 = \lim_{n\to\infty} \sup(a_n)$.

c) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $\{1, -1, 2, -2, 3, -3 \ldots\}$.
Then $\lim_{n\to\infty} \inf a_n = -\infty < \infty = \lim_{n\to\infty} \sup a_n$.

d) Let $a_n = \frac{(-1)^n}{1+\frac{1}{n}}$. Then $\lim_{n\to\infty} \inf a_n = -1$ while $\lim_{n\to\infty} \sup a_n = 1$.

SOME SPECIAL SEQUENCES

We shall now compute the limits of some sequences which occur frequently, but before doing so let us review the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \text{where} \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Example:

a) $(x+y)^2 = y^2\binom{2}{0} + xy\binom{2}{1} + x^2\binom{2}{2} = y^2\,\frac{2!}{0!\cdot 2!} + xy\,\frac{2!}{1!\cdot 1!} + x^2\,\frac{2!}{2!\cdot 0!} = y^2 + 2\,xy + x^2.$

b) $$(x+y)^3 = y^3\binom{3}{0} + xy^2\binom{3}{1} + x^2\,y\binom{3}{2} + x^3\binom{3}{3}$$
$$= y^3\,\frac{3!}{0!\cdot 3!} + xy^2\,\frac{3!}{1!\cdot 2!} + x^2\,y\,\frac{3!}{2!\cdot 1!} + x^3\,\frac{3!}{3!\cdot 0!}$$
$$= y^3 + 3\,xy^2 + 3\,x^2\,y + x^3\,.$$

Note: To proceed we will also need the following: If $0 \le x_n \le s_n$ for some $n \ge N$, where N is a fixed number, and if $s_n \to 0$, then it is always true that $x_n \to 0$.

• Theorem:

a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof:

a) Take $n > \left(\frac{1}{\varepsilon}\right)^{1/p}$. (Note that the Archimedean property of $\mathbb{R}$ is used here). ✓

b) If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ and by the binomial theorem we have
$$1 + n\,x_n \le (1+x_n)^n = p \implies 0 < x_n \le \frac{p-1}{n}$$
Hence $x_n \to 0$.
If $p = 1$, b) is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals. ✓

c) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \ge 0$ and by the binomial theorem we have
$$n = (1+x_n)^n \ge \frac{n\,(n-1)}{2}\,x_n^2 \implies 0 \le x_n \le \sqrt{\frac{2}{n-1}} \qquad (n \ge 2) \qquad ✓$$

d) Let $k > \alpha$ and $k > 0$. For $n > 2\,k$, observe that
$$n - k + 1 - \frac{n}{2} = \frac{n}{2} - k + 1 > k - k + 1 = 1 > 0.$$
Thus, $n - k + 1 > \frac{n}{2}$.

Notice that

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!}\, p^k > \frac{n^k}{2^k\, k!}\, p^k \quad \text{(why?)}$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k\, k!}{p^k}\, n^{\alpha-k} \quad (n > 2\, k)$$

Since $\alpha - k < 0$, it follows that $n^{\alpha-k} \to 0$ by a) . ✓

e) If Simply take $\alpha = 0$ in d) . ✓ ■

SERIES

In the remainder of this chapter all sequences and series under consideration will be complex-valued, unless the contrary is explicitly stated.

The Cauchy criterion (i.e. every convergent sequence is Cauchy) can be restated in the following form:

• Theorem:

$\sum a_n$ converges iff for every $\varepsilon > 0$ there is an integer N such that $\left|\sum_{k=n}^{m} a_k\right| \le \varepsilon$ if $m \ge n \ge N$.

** In particular, by taking $m = n$, the above expression becomes $|a_n| \le \varepsilon$. Also notice that if $n = N$ and $m \to \infty$, the expression becomes $\left|\sum_{n=N}^{\infty} a_n\right| \le \varepsilon$.**

Proof:
In $\mathbb{R}$ and in $\mathbb{C}$ every Cauchy sequence converges (why?). Thus the sequence s_n of partial sums is convergent iff it is Cauchy. Now, s_n is Cauchy if for any $\varepsilon > 0$ there is some N such that $n, m \ge N$ implies $|s_n - s_m| < \varepsilon$.
If $m \ge n$ we have

$$|s_n - s_m| = |s_m - s_n| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{m} a_k\right|.$$ ■

• Corollary:
If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of $\sum a_n$. For instance the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, as we'll demonstrate later.

• Corollary:
A series of nonnegative terms converges iff its partial sums form a bounded sequence.

We now turn to a convergence test of a different nature, the so-called comparison test:

• Theorem:
a) If $|a_n| \leq c_n$ for $n \geq \mathcal{N}_0$, where $\mathcal{N}_0$ is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.
b) If $a_n \geq d_n \geq 0$ for $n \geq \mathcal{N}_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.
Note that b) applies only to series of nonnegative terms a_n.

Proof:
Given $\varepsilon > 0$, there exists $\mathcal{N} \geq \mathcal{N}_0$ such that $m \geq n \geq \mathcal{N}$ implies $\sum_{k=n}^{m} c_k \leq \varepsilon$ by the Cauchy criterion.
Hence

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \varepsilon$$

and a) follows.
Next, b) follows from a), for if $\sum a_n$ converges, so must $\sum d_n$. ■

Note: The comparison test is a very useful one. To use it efficiently though, we have to become familiar with a number of series of nonnegative terms whose convergence or divergence is known.

SERIES OF NONNEGATIVE TERMS

The simplest of all is perhaps the geometric series:

• Theorem:
If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Proof:
If $x \neq 1$,

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

The result follows if we let $n \to \infty$.

For $x = 1$, we get $1 + 1 + 1 + \ldots + 1$, which evidently diverges. ■

In many cases which occur in applications, the terms of the series decrease monotonically. The following theorem of Cauchy is therefore of particular interest. The striking feature of the theorem is that a rather "thin" subsequence of $\{a_n\}$ determines the convergence or divergence of $\sum a_n$.

• Theorem:

Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff the series $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots$ converges.

Proof:

Since the series under consideration has nonnegative terms, it suffices to consider boundedness of the partial sums.

Let $s_n = a_1 + a_2 + \ldots + a_n$ and $t_k = a_1 + 2 a_2 + \ldots + 2^k a_{2^k}$.

For $n < 2^k$,

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k, \end{aligned}$$

so that

$$s_n \leq t_k. \tag{8}$$

On the other hand, if $n > 2^k$,

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \tfrac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ &= \tfrac{1}{2}t_k, \end{aligned}$$

so that

$$2s_n \geq t_k. \tag{9}$$

By (8) and (9), the sequences $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded. This completes the proof.

■

• Corollary 1:

$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof:

If $p \leq 0$, $\lim_{n \to \infty} \frac{1}{n^p} = \lim_{n \to \infty} n^{-p} = \infty$ and the series diverges.

If $p > 0$, the sequence $\frac{1}{n^p}$ decreases and the above theorem applies and we are led to the series $\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}$. Now, $2^{1-p} < 1$ iff $1 - p < 0$, and the result follows by comparison with the geometric series $\sum x^k$, where $x = 2^{1-p}$. ■

• Corollary 2:

If $p > 1$, $\sum \frac{1}{n(\log n)^p}$ converges. If $p \leq 1$, the series diverges .

Proof:

The monotonicity of the logarithmic function implies that $\{\log n\}$ increases. Hence $\left\{\frac{1}{n \log n}\right\}$ decreases, and we can apply the above theorem.

This leads us to the series

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \left(\log 2^k\right)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

and the conclusion follows. ■

This procedure may evidently be continued. For instance, $\sum_{n=3}^{\infty} \frac{1}{n \log n \cdot \log(\log n)}$ diverges, whereas $\sum_{n=3}^{\infty} \frac{1}{n \log n \cdot (\log(\log n))^2}$ converges.

THE NUMBER e

Definition: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Since
$$s_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots + \frac{1}{1 \cdot 2 \cdot \ldots \cdot n}$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} < 3$$

the series converges, and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy.

It is of interest to note that e can also be defined by means of another limit process; the proof provides a good illustration of operations with limits.

• Theorem:

$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Proof:

Let $s_n = \sum_{k=0}^{\infty} \frac{1}{k\,!}$ and $t_n = \left(1 + \frac{1}{n}\right)^n$.

Clearly the sequence s_n is monotonically increasing. To see that t_n is also monotonically increasing, observe that by the binomial theorem $t_n = \left(1 + \frac{1}{n}\right)^n > \left(1 + n\,\frac{1}{n}\right) = 2$. In fact, if $a > -1,\, a \neq 0$, then $(1 + a)^n > 1 + n\,a$.

Now we have

$$\begin{aligned}\frac{t_{n+1}}{t_n} &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = \left(1 + \frac{1}{n}\right)\left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1} \\ &= \left(1 + \frac{1}{n}\right)\left(\frac{n^2 + 2\,n}{(n+1)^2}\right)^{n+1} \\ &= \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \\ &> \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n+1}\right) = 1\end{aligned}$$

Thus $t_{n+1} > t_n$ and t_n is increasing, as desired.

By the binomial theorem,

$t_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right) \leq s_n < e$.

Thus, $\{t_n\}_{n=1}^{\infty}$ is also a bounded sequence. Hence $\lim_{n\to\infty} t_n = \alpha \leq e$.

Next, if $n \geq m$,

$$\begin{aligned}t_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{m-1}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right) \\ &\geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{m-1}{n}\right)\end{aligned}$$

Thus,

$$\alpha > t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{m-1}{n}\right)$$

Now let $n \to \infty$, keeping m fixed. We get

$$\alpha > 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{m!} = s_m$$

This means that $\alpha \geq \lim_{n\to\infty} s_m = e$, which implies that $\alpha = e$. ∎

The rapidity with which the series $\sum \frac{1}{n!}$ converges can be estimated as follows:

If s_n has the same meaning as above, we have

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots < \frac{1}{(n+1)!}\left(1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \ldots\right) = \frac{1}{n!\,n}$$

so that $0 < e - s_n < \frac{1}{n!\,n}$.

Thus, s_{10}, for instance, approximates e with an error less than 10^{-7}. The inequality is of theoretical interest as well, since it enables us to prove the irrationality of e.

• Theorem:
e is irrational.

Proof:
Suppose e is rational. Then $e = \frac{p}{q}$, where p and q are positive integers. By the inequality preceding this theorem, we have

$$(*) \qquad 0 < q!\,(e - s_q) < \frac{1}{q}.$$

By our assumption, $q!\,e$ is an integer. Since

$$q!\,s_q = q!\left(1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right)$$

is an integer, we see that $q!\,(e - s_q)$ is an integer.
But then since $q \geq 1$, (*) implies the existence of an integer between 0 and 1. (⇒⇐)
Thus we have reached a contradiction and therefore we conclude thet e must be irrational. ■

THE ROOT AND RATIO TESTS

• Theorem (Root Test):
Given $\sum a_n$, put $\alpha = \lim\limits_{n\to\infty} \sup \sqrt[n]{|a_n|}$.

Then,
a) If $\alpha < 1$, $\sum a_n$ converges.
b) If $\alpha > 1$, $\sum a_n$ diverges.
c) If $\alpha = 1$, the test gives no information.

Proof:
a) If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$. That is, $n \geq N$ implies $|a_n| < \beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges. Convergence of $\sum a_n$ follows now from the comparison test. ✓

b) If $\alpha > 1$, then, again, there is a sequence $\{n_k\}$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Hence $|a_n| > 1$ for infinitely many values of n, so that the condition $a_n \to 0$, necessary for the convergence of $\sum a_n$, does not hold. ✓

c) Consider the series $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$.

We can see that $\sqrt[n]{1/n} \to 1$ and $\sqrt[n]{1/n^2} \to 1$ but the former series diverges while the latter converges. Hence, we have that $\alpha = 1$ gives no information. ✓ ■

• Theorem (Ratio Test):
The series $\sum a_n$
a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
b) diverges. if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Proof:
a) If condition a) holds, we can find $\beta < 1$, and an integer N such that $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for $n \geq N$. In particular,

$$\begin{aligned}
&|a_{N+1}| < \beta |a_N|, \\
&|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&|a_{N+p}| < \beta^p |a_N|.
\end{aligned}$$

That is, $|a_{N+p}| < |a_n| \beta^{-N} \beta^n$ for $n \geq N$, and a) follows from the comparison test, since $\sum \beta^n$ converges. ✓

b) If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, it is easy to see that the condition $a_n \to 0$ does not hold, and a) follows. ✓ ■

Note: The knowledge that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ implies nothing about the convergence of $\sum a_n$. The series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ demonstrate this.

Example:
a) Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$$

for which

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}}$$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}}$$

$$\lim_{n\to\infty} \sup \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = \infty \quad .$$

The root test indicates convergence whereas the ratio test does not apply.

b) The same is true for the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \ldots$$

where

$$\lim_{n\to\infty} \inf \frac{a_{n+1}}{a_n} = \frac{1}{8}$$

$$\lim_{n\to\infty} \sup \frac{a_{n+1}}{a_n} = 2$$

but

$$\lim_{n\to\infty} \inf \sqrt[n]{a_n} = \frac{1}{2} \, .$$

• Theorem:

For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof:

We shall prove the second inequality; the proof of the first is quite similar.

Put

$$\alpha = \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that $\frac{c_{n+1}}{c_n} \leq \beta$ for $n \geq N$. In particular, for any $p > 0$,

$$c_{N+k+1} \leq \beta \, c_{N+k} \quad (k = 0,\ 1,\ \ldots,\ p-1)\,.$$

Multiplying these inequalities, we obtain

$$c_{N+p} \leq \beta^p \, c_N \, .$$

or

$$c_n \leq c_N \, \beta^{-N} \cdot \beta^n \qquad (n \geq N)\,.$$

Hence

$$\sqrt[n]{c_n} \leq \sqrt[n]{c_N \, \beta^{-N}} \cdot \beta \, ,$$

so that

$$\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \beta$$

because $\lim_{n\to\infty} \sup \sqrt[n]{c_N \, \beta^{-N}} = 1$.

Since $\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \beta$ for every $\beta > \alpha$, we have $\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \alpha$. ■

POWER SERIES

Definition: Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series.

The numbers c_n are called the coefficients of the series, while z is a complex number.

In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle (the circle of convergence), such that the power series converges if z is in the interior of the circle and diverges otherwise (to cover all cases we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior of the circle of convergence is much more varied and cannot be described so simply.

• Theorem:

Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \quad , R = \frac{1}{\alpha}$$

(If $\alpha = 0$, $R = +\infty$. If $\alpha = +\infty$, $R = 0$)

Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Proof:

Put $a_n = c_n z^n$ and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R} \qquad \blacksquare$$

Note: R is called the convergence radius of $\sum c_n z^n$.

Example:

a) The series $\sum n^n z^n$ has $R = 0$.

b) The series $\sum \frac{z^n}{n!}$ has $R = \infty$ (In this particular case the ratio test is easier to apply than the root test).

c) The series $\sum z^n$ has $R = 1$. If $|z| = 1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n \to \infty$.

d) The series $\sum \frac{z^n}{n}$ has $R = 1$. It diverges if $z = 1$ and converges for all other z with $|z| = 1$ (the last assertion will be proved later).

e) The series $\sum \frac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$ by the comparison test, since $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$.

SUMMATION BY PARTS

• Theorem:

Given two sequences $\{a_n\}$, $\{b_n\}$, put $A_n = \sum_{k=0}^{n} a_k$ if $n \geq 0$; put $A_{-1} = 0$.

Then, if $0 \leq p \leq q$, we have

$$(**) \qquad \sum_{n=p}^{q} a_n\, b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q\, b_q - A_{p-1}\, b_p\,.$$

Proof:

$$\sum_{n=p}^{q} a_n\, b_n = \sum_{n=p}^{q} (A_n - A_{n-1})\, b_n = \sum_{n=p}^{q} A_n\, b_n - \sum_{n=p}^{q} A_{n-1}\, b_n$$

$$= \sum_{n=p}^{q} A_n\, b_n - \sum_{n=p-1}^{q-1} A_n\, b_{n+1}$$

$$= A_q\, b_q + \sum_{n=p}^{q-1} A_n\, b_n - \sum_{n=p}^{q-1} A_n\, b_{n+1} - A_{p-1}\, b_p$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q\, b_q - A_{p-1}\, b_p. \qquad \blacksquare$$

Note: Formula (**) from the above theorem, the so-called partial summation formula, is useful in the investigation of series of the form $\sum a_n\, b_n$, particularly when $\{b_n\}$ is monotonic.

• Theorem:

Suppose

i) The partial sums A_n of $\sum a_n$ form a bounded sequence.

ii) $b_0 \geq b_1 \geq b_2 \geq \ldots \geq 0$.

iii) $\lim_{n \to \infty} b_n = 0$.

Then, if all three properties hold, $\sum a_n\, b_n$ converges.

Proof:

Choose M such that $|A_n| \leq M$ for all n. Given $\varepsilon > 0$, there is an integer $\mathcal{N}$ such that $b_{\mathcal{N}} \leq \frac{\varepsilon}{2M}$. For $\mathcal{N} \leq p \leq q$, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M\left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2\,M\,b_p \leq 2\,M\,b_{\mathcal{N}} \leq \varepsilon.\end{aligned}$$

Convergence then follows from the Cauchy criterion. ■

• Theorem:

Suppose

i) $|c_1| \geq |c_2| \geq |c_3| \geq \ldots.$

ii) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(m = 1, 2, 3, \ldots)$

** Series for which ii) holds are called “alternating series” **

iii) $\lim_{n\to\infty} c_n = 0.$

Then, if all three properties hold, $\sum c_n$ converges.

Proof:

Put $\sum c_n = \sum a_n b_n$, where $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

Notice that $A_n = \sum_{k=1}^{n} a_n$ is bounded with $|A_n| \leq 1$ and $b_n \geq b_{n+1}$. Thus the result follows from the previous theorem. ■

• Theorem:

Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \ldots$ and $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

Proof:

Put $a_n = z^n$, $b_n = c_n$. Then

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1-z^{n+1}}{1-z}\right| \leq \frac{|1| + |z|^{n+1}}{|1-z|} = \frac{|1| + |1|}{|1-z|} = \frac{2}{|1-z|}$$

if $|z| = 1$, $z \neq 1$. ■

ABSOLUTE CONVERGENCE

The series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges.

• Theorem:
If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof:
The assertion follows from the inequality $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} a_k$, plus the Cauchy criterion. ■

Note: For series of positive terms, absolute convergence is the same as convergence.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges non-absolutely.
For instance, the series

$$\sum \frac{(-1)^n}{n}$$

converges non-absolutely.

The comparison test, as well as the root and ratio tests, are tests for absolute convergence and therefore cannot give any information about non-absolutely convergent series. Summation by parts can sometimes be used to handle the latter. In particular, power series converge absolutely in the interior of the circle of convergence.
We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. However, for non-absolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

ADDITION AND MULTIPLICATION OF SERIES

• Theorem:
If $\sum a_n = A$, and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$, and $\sum c\, a_n = cA$, for any fixed c.

Proof:
Let $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$. Then $A_n + B_n = \sum_{k=0}^{n} (a_k + b_k)$.
Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, we see that $\lim_{n\to\infty} (A_n + B_n) = A + B$.

The proof of the second assertion is similar. ■

Thus, two convergent series may be added term by term, and the resulting series converges to the sum of the two series. The situation becomes more complicated when we consider multiplication of two series.
To begin with, we have to define the product. This can be done in several ways; we shall now consider the so-called Cauchy product:

Definition: Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k} \quad (n = 0,\ 1,\ 2,\ \ldots)$$

and call $\sum c_n$ the product of the two given series.

This definition may be motivated as follows. If we take two power series $\sum a_n z^n$ and $\sum b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \ldots) \cdot (b_0 + b_1 z + b_2 z^2 + \ldots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \ldots \\ &= c_0 + c_1 z + c_2 z^2 + \ldots\end{aligned}$$

Now by setting $z = 1$, we arrive at the above definition.

Example:

If $\quad A_n = \sum_{k=0}^{n} a_k\,, \qquad B_n = \sum_{k=0}^{n} b_k\,, \qquad C_n = \sum_{k=0}^{n} c_k$

and $A_n \to A,\ B_n \to B$, then it is not at all clear that $\{c_n\}_{n=1}^{\infty}$ will converge to AB, since we do not have $C_n = A_n B_n$.
The dependence of $\{c_n\}$ on $\{A_n\}$ and $\{B_n\}$ is quite a complicated one. We shall now see that the product of two convergent series may actually diverge.
The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

converges by the alternating test.

We form the product of this series with itself and obtain

$$\sum_{n=0}^{\infty} c_n = 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\,\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\,\sqrt{2}} + \frac{1}{\sqrt{2}\,\sqrt{3}} + \frac{1}{\sqrt{4}}\right) \ldots$$

where $c_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\ \sqrt{n-k+1}}$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)\,(n-k+1)}}$$

To estimate $|c_n|$, observe that the function

$$f(x) = (x+1)\,(n-x+1) \qquad x \in [0,\ n]$$

is differentiable over $(0,\ n)$ with derivative

$$f\,'(x) = (n-x+1) - (x+1) = n - 2\,x = 0 \qquad \text{when } x = \frac{n}{2}.$$

Notice that $f\,''(x) = -2 < 0 \Longrightarrow f\,''\left(\frac{n}{2}\right) < 0$, implying that $f\left(\frac{n}{2}\right)$ is a local maximum.

Also notice that

$$f\left(\frac{n}{2}\right) = \left(\frac{n}{2}+1\right)\left(n-\frac{n}{2}+1\right) = \left(\frac{n}{2}+1\right)^2 > \frac{n}{2}+1$$

whereas

$$f(n) = f(0) = n+1 < \frac{n^2}{4} + n + 1 = \left(\frac{n}{2}+1\right)^2 = f\left(\frac{n}{2}\right)$$

Thus $f\left(\frac{n}{2}\right)$ is the absolute maximum. In particular, for $0 \le k \le n$ we have

$$(k+1)\,(n-k+1) \le \left(\frac{n}{2}+1\right)^2 \quad \text{or} \quad \sqrt{(k+1\,(n-k+1)} \le \sqrt{\left(\frac{n}{2}+1\right)^2} = \frac{n}{2}+1.$$

Therefore

$$\frac{1}{\sqrt{(k+1)\,(n-k+1)}} \ge \frac{1}{\frac{n}{2}+1} = \frac{2}{n+2}\ .$$

Thus

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)\,(n-k+1)}} \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2\,(n+1)}{n+2}$$

which implies that

$$\limsup_{n\to\infty} |c_n| \ge \lim_{n\to\infty}\ \sup \frac{2\,(n+1)}{n+2} = 2$$

suggesting that $\sum c_n$ diverges by he divergence test.

In view of the next theorem, due to Mertens, we note that we have here considered the product of two nonabsolutely convergent series.

• Theorem:

Suppose

i) $\sum_{n=0}^{\infty} a_n$ converges absolutely .

ii) $\sum_{n=0}^{\infty} a_n = A$

iii) $\sum_{n=0}^{\infty} b_n = B$

iv) $c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$ $(n = 0,\ 1,\ 2,\ \ldots)$

Then, if all four conditions are satisfied we have that $\sum_{n=0}^{\infty} c_n = A\,B$.

** That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absoultely. **

Proof:

Put $\quad A_n = \sum_{k=0}^{n} a_k\ ,\quad B_n = \sum_{k=0}^{n} b_k\ ,\quad C_n = \sum_{k=0}^{n} c_k\ ,\quad \beta_n = B_n - B$.

Then,

$$\begin{aligned} C_n &= a_0\, b_0 + (a_0\, b_1 + a_1\, b_0) + \ldots + (a_0\, b_n + a_1\, b_{n-1} + \ldots + a_n\, b_0) \\ &= a_0\, \beta_n + a_1\, \beta_{n-1} + \ldots + a_n\, \beta_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \ldots + a_n (B + \beta_0) \\ &= A_n\, B + a_0\, \beta_n + a_1\, \beta_{n-1} + \ldots + a_n\, \beta_0 \end{aligned}$$

Put

$$\gamma_n = a_0\, \beta_n + a_1\, \beta_{n-1} + \ldots + a_n\, \beta_0.$$

We wish to show that $C_n \to AB$. Since $A_n\, B \to AB$, it suffices to show that $\lim_{n\to\infty} \gamma_n = 0$.

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n|.$$

(It is here that we use i)). Let $\varepsilon > 0$ be given. By iii), $\beta_n \to 0$. Hence we can choose $\mathcal{N}$ such that $|\beta_n| \le \varepsilon$ for $n \ge \mathcal{N}$, in which case

$$\begin{aligned} |\gamma_n| &\le |\beta_0\, a_n + \ldots + \beta_{\mathcal{N}}\, a_{n-\mathcal{N}}| + |\beta_{\mathcal{N}+1}\, a_{n-\mathcal{N}-1} + \ldots + \beta_n\, a_0| \\ &\le |\beta_0\, a_n + \ldots + \beta_{\mathcal{N}}\, a_{n-\mathcal{N}}| + \varepsilon\, \alpha\ . \end{aligned}$$

Keeping $\mathcal{N}$ fixed, and letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sup |\gamma_n| \le \lim_{n\to\infty} \sup |\beta_0\, a_n + \ldots + \beta_{\mathcal{N}}\, a_{n-\mathcal{N}}| + \varepsilon\, \alpha = \varepsilon\, \alpha\ .$$

Since ε is arbitrary, $c_n \to AB$ (because $\gamma_n \to 0$) as desired. ■